

A (Bad, Incomplete) Translation of ‘*Sur des variétés riemanniennes à groupe d’holonomie G_2 ou $\text{Spin}(7)$* ’ by E. Bonan, with Additions by the Translator

Sam Close

This is a bad translation¹ of Bonan’s seminal work [1], with some unsolicited comments from the translator. The best publicly available English translation (that I could easily find) prior to writing this Robert Bryant’s summary of the paper as an answer to a MathOverflow question [2], which does contain all the relevant information about what is discussed, but is not itself a translation.

Changes Some changes and choices were made. Some notation has been changed or adapted to modern styles.

- Double-struck font is used for the reals and integers, instead of bold font.
- $\times$ notates the antisymmetric pairing of two vectors in 7-dimensions, as it appears in **1.**, as opposed to a period.
- The additional basis vector defined in **2.** is notated e_7 , not $e_{7'}$.
- Every Lie group is upright, e.g. $\text{Spin}(7)$, not $\textit{Spin}(7)$.

There are some typographical errors in the original.

- There are a number of lines which seem like they should end, but instead carry on via a comma.
- Equation (3) does not exist.
- Reference 5 (A. Lichnerowicz, Théorie globale des connexions, 1954) is never actually cited in the text.
- In corollary 2, the expression reads $\varphi_p = \gamma \wedge \mu_{p-6} + \mu_p$ whereas it should be μ_{p-4} .

Most notably, Bonan does not prove any of his propositions or corollaries: thus the proofs added are my own contribution.

The translation is structured as it appears in the original, albeit with white-space on the right hand side of the page for translator comments. Additionally, citations are not necessarily numbered the same as in the original. More detail has also been added to the bibliography.

¹This is translated by me, my French knowledge, an online French to English dictionary, and, for some strange phrases, an online translator. No LLMs were used.

References

- [1] Edmond Bonan. “Sur des variétés riemanniennes à groupe d’holonomie G_2 ou $Spin(7)$ ”. French. In: *Comptes rendus hebdomadaires des séances de l’Académie des Sciences. Série A* 262 (1966). The original can be accessed here., pp. 127–129 (cit. on p. i).
- [2] Robert Bryant. *English translation of paper: Sur les variétés riemanniennes à groupe d’holonomie G_2 ou $Spin(7)$* . The title is the title of the original post. This reference is referring to Robert Bryant’s comment. URL: <https://mathoverflow.net/a/298290> (cit. on p. i).

On Riemannian Manifolds with Holonomy Group G_2 or $\text{Spin}(7)$

Edmond Bonan

Abstract

This completes work that was done in [2]: G_2 leaves invariant a 3-form and a 4-form; $\text{Spin}(7)$ leaves invariant a 4-form. Applications to holonomy groups.

1. Let $(1, e_i)$, $i \in \mathbb{Z}_7$ be a basis for the octonions: every triple

$$(e_i, e_{i+1}, e_{i+3})$$

forms a basis for the quaternions.

The group $\text{SO}(7)$ acts* on $\mathbb{R}^7$

$$\mathbb{R}^7 = \bigoplus_{i \in \mathbb{Z}_7} \mathbb{R}e_i.$$

Let G_2 be the largest subgroup of $\text{SO}(7)$ which preserves the vector valued 2-form T ,[†]

$$X, Y \in \mathbb{R}^7 \rightarrow T(X, Y) = X \times Y \in \mathbb{R}^7; \quad X \times Y = -Y \times X.$$

Additionally, $\mathbb{R}^7$ is equipped with a canonical scalar product

$$X, Y \in \mathbb{R}^7 \rightarrow (X, Y) = \sum_{i \in \mathbb{Z}_7} X^i Y^i.$$

Then define the trilinear form α

$$X, Y, Z \in \mathbb{R}^7 \rightarrow \alpha(X, Y, Z) = (X, Y \times Z)$$

which is manifestly totally antisymmetric. If we denote (ω^i) the dual basis of (e_i) , it immediately follows that

$$\alpha = \sum_{i \in \mathbb{Z}_7} \omega^i \wedge \omega^{i+1} \wedge \omega^{i+3}. \quad (1)$$

We deduce that the 4-form β is adjoint to the 3-form α :

$$\beta = \star \alpha = \sum_{i \in \mathbb{Z}_7} \omega^i \wedge \omega^{i+1} \wedge \omega^{i+2} \wedge \omega^{i+5}. \quad (2)$$

Proposition 1. G_2 leaves α and β invariant.

Proof. It is easy to show that α is G_2 invariant. G_2 invariance of T means

$$T(X, Y) = g^{-1}T(gX, gY).$$

* This sentence is difficult to translate in a way that makes sense, because it may not make sense. In particular, no action is defined: only $\mathbb{R}^7$ is defined. Charitably, this could be read as ‘we define $\mathbb{R}^7$, given as $\bigoplus_{i \in \mathbb{Z}_7} \mathbb{R}e_i$, to be the fundamental/defining representation of $\mathbb{R}^7$ ’.

[†] Two remarks here. Firstly, note that the use of G_2 here is suggestive notation, rather than being provably true from the data given: Bonan does not prove that the group that preserves T is actually G_2 as we know it. Secondly, we took quite a liberty with the translation of the latter part of this sentence. The original says that G_2 preserves ‘*la table de multiplication T*’, which can only be fairly translated as ‘the multiplication table T ’. This sounds a bit odd in English. What the sentence is actually telling us is that T is G_2 invariant, so we have changed the translation to reflect this.

Thus,

$$\begin{aligned}(g^*\alpha)(X, Y, Z) &= \alpha(gX, gY, gZ) = (gX, T(gY, gZ)) \\ &= (gX, g^{-1}T(Y, Z)) = (X, T(Y, Z)) = \alpha(X, Y, Z).\end{aligned}$$

The Hodge dual is equivariant with respect to the natural $G_2 < SO(7)$ action, thus $g^*\beta = g^* \star \alpha = \star(g^*\alpha) = \star\alpha = \beta$. $\square$

We have shown by direct calculation* that for all 2-forms φ_2 :

$$\alpha \wedge \varphi_2 = 0 \iff \varphi_2 = 0.$$

It follows that if we denote φ_p a p -form:

Proposition 2. *Provided that $p \leq 2$: $\alpha \wedge \varphi_p = 0$ implies $\varphi_p = 0$.*

Proof. $p = 0$ is trivial. $p = 2$ is already ‘proven’. For the final case, $p = 1$, it is easiest to verify explicitly that the only solution is $\varphi_1 = 0$. $\square$

Corollary 1. *For $p \leq 5$, every p -form φ_p admits a unique direct sum decomposition of the form*

$$\varphi_p = \alpha \wedge \mu_{p-3} + \mu_p \text{ where } \alpha \wedge \star\mu_\bullet = 0.$$

Proof. For $p = 0, 1, 2$ this is trivially true. For $p = 3$, we have the natural splitting of 3-forms into $\mathbb{R}\alpha$ and its complement. This immediately realises the decomposition $\varphi_3 = \mu_0\alpha + \mu_3$. Since we have a metric, we can force α to be orthogonal to μ_3 , hence $\alpha \wedge \star\mu_3 = 0$. For $p = 4$ we can perform the same trick: we know that $\mu_1 \wedge \alpha = 0 \iff \mu_1 = 0$ from proposition 2 so $\mu_1 \wedge \alpha$ can serve as our canonical 4-form to decompose the space into

$$\varphi_4 = \mu_1 \wedge \alpha + \mu_4.$$

This decomposition can be made orthogonal since we have a metric. The same proof holds, *mutatis mutandis*, for $p = 5$. $\square$

The Lie algebra of G_2 can be realised as a collection of 7×7 matrices $A = (A_{ij})$ satisfying the relations

$$A_{ij} + A_{ji} = 0; \quad A_{i+1,i+3} + A_{i+4,i+5} + A_{i+2,i+6} = 0.$$

We remark on the three triples:

$$(e_i, e_{i+1}, e_{i+2}), (e_{i+4}, e_{i+5}, e_i), (e_{i+6}, e_i, e_{i+2})$$

each form a basis for the quaternions[†].

* This is not shown in the original text: he may be referring to earlier work or work he did but never published. It is not difficult for one to check that indeed this is true.

[†] A point of curiosity: it’s not clear why Bonan makes this (or the previous) remark about the quaternions. It is completely unrelated to the rest of the paper.

2. Let $\text{Spin}(7) \subset \text{SO}(8)$ act* on $\mathbb{R}^8$

$$\mathbb{R}^8 = \mathbb{R}e_7 \oplus \bigoplus_{i \in \mathbb{Z}_7} \mathbb{R}e_i.$$

The previous group G_2 is the isotropy group of the vector e_7 and acts in the 7-plane defined by the equation $\omega^7 = 0$ where we denote (ω^α) the dual basis of (e_α) ($\alpha \in \{7\} \cup \mathbb{Z}_7$).

The Lie algebra[†] of $\text{Spin}(7)$ can be realised as a collection of matrices $A = (A_{\alpha\beta})$ satisfying the relations

$$\begin{cases} A_{\alpha\beta} + A_{\beta\alpha} = 0; \\ A_{7i} + A_{i+1,i+3} + A_{i+4,i+5} + A_{i+2,i+6} = 0. \end{cases} \quad (3)$$

Define the 4-form γ

$$\gamma = \omega^7 \wedge \alpha' + \beta'$$

where α' and β' are the obvious extensions of α and β to $\mathbb{R}^8$

$$\alpha' = \alpha \circ P, \quad \beta' = \beta \circ P$$

where P denotes the orthogonal projector to $\mathbb{R}^7$,

$$\gamma = \sum_{i \in \mathbb{Z}_7} \omega^7 \wedge \omega^i \wedge \omega^{i+1} \wedge \omega^{i+3} + \sum_{i \in \mathbb{Z}_7} \omega^i \wedge \omega^{i+1} \wedge \omega^{i+2} \wedge \omega^{i+5}. \quad (4)$$

Proposition 3. γ is invariant under $\text{Spin}(7)^\ddagger$.

Proposition 4. Provided that $p \leq 2$: $\gamma \wedge \varphi_p = 0$ implies $\varphi_p = 0$.

Proof. As with the proof of proposition 2, $p = 0$ is obvious. The proof for $p = 1, 2$ can be shown directly. $\square$

Corollary 2. For $p \leq 6$, any p -form φ_p admits a unique direct sum decomposition of the type

$$\varphi_p = \gamma \wedge \mu_{p-4} + \mu_p, \quad \text{where } \gamma \wedge \star \mu_\bullet = 0.$$

Proof. The proof proceeds, *mutatis mutandis*, the same as corollary 1. $\square$

3. Let G be a closed subgroup of the orthogonal group $O(d)$.

Definition 1. We call V_d , a Riemannian manifold of dimension d , a G -manifold if the holonomy group is a subgroup of G . The frame bundle $E[V_d, O(d)]$ admits a subbundle $E^a[V_d, G]$.[§]

The previous results apply at every point $x \in V_d$: in particular.

Proposition 5. All G_2 -manifolds V_7 admit a global 3-form and 4-form which are covariantly constant.

Proposition 6. All $\text{Spin}(7)$ -manifolds V_8 admit a global 4-form which is covariantly constant.

* As in 1., this is a strange sentence, though arguably stranger than before. The original reads ‘[f]aisons opérer $\text{Spin}(7) \subset \text{SO}(8)$ dans $\mathbb{R}^8$ ’ which a translator will tell you means (literally) something like ‘[L]et’s define $\text{Spin}(7) \subset \text{SO}(8)$ on $\mathbb{R}^8$ ’. Once again, no action is defined, so we must assume that the action is implied to be the fundamental representation.

† Bonan says, incorrectly, that these matrices are the matrices of $\text{Spin}(7)$ the Lie group. These are absolutely the matrices of $\mathfrak{spin}(7)$.

‡ This is one of the central results of this paper, yet it is very much not obvious how one proves it given the preceding discussion.

§ It is not clear if there is meaning behind the superscript a beyond clarifying that this is a subbundle.

Proof. Both proposition 5 and 6 follow from the holonomy principle (cf. [5]) $\square$

The established decompositions in corollaries 1 and 2 are evidently global.

In manipulating Betti numbers $b_k(V_d)$ it follows from propositions 5 and 6, assuming V_d is compact:

$$b_3(V_7) \neq 0; \quad b_3(V_7) \geq b_1(V_7); \quad b_4(V_8) \neq 0; \quad b_4(V_8) \geq b_1(V_8).$$

Let us now define R_{ijkl} the Riemann curvature tensor. With respect to the frame bundle $E^a[V_7, G_2]$ the curvature satisfies the additional relations:

$$R_{i+1, i+3, kl} + R_{i+4, i+5, kl} + R_{i+2, i+6, kl} = 0, \quad \forall k, l, i \in \mathbb{Z}_7. \quad (5)$$

Calculating the Ricci tensor,

$$R_{ij} = \sum_{l \in \mathbb{Z}_7} R_{iljl}. \quad (6)$$

It suffices to calculate R_{00} and R_{10} :

$$\begin{aligned} R_{00} &= R_{0101} + R_{0202} + R_{0303} + R_{0404} + R_{0505} + R_{0606}; \\ R_{01} &= R_{0212} + R_{0313} + R_{0414} + R_{0515} + R_{0616}. \end{aligned}$$

Using the relations (5) and noting $R_{ijkl} = R_{klij}$:

$$\begin{aligned} R_{00} &= R_{0125} + R_{0164} + R_{0243} + R_{0251} + R_{0324} + R_{0356} \\ &\quad + R_{0416} + R_{0432} + R_{0512} + R_{0563} + R_{0635} + R_{0641}; \\ R_{01} &= R_{0205} + R_{0236} + R_{0354} + R_{0362} + R_{0435} \\ &\quad + R_{0460} + R_{0543} + R_{0520} + R_{0623} + R_{0604}. \end{aligned}$$

These two expressions vanish by virtue of the algebraic Bianchi identity*.

Proposition 7. *All G_2 -manifolds V_7 are Ricci flat.*

Proof. The only additional work required here is convincing yourself that it is sufficient to check only R_{00} and R_{01} : or checking all 28 components. $\square$

Likewise:

Proposition 8. *All $\text{Spin}(7)$ -manifolds V_8 are Ricci flat.*

Proof. After taking into account the extra dimensions, and the identities (3), the proof proceeds *mutatis mutandis* the same as with G_2 -manifolds. $\square$

* Bonan calls these the Ricci identities. Nowadays, Ricci identity refers to the relationship between the covariant derivative and the curvature tensor.

4. Remarks

- After the work done in [3] by B. Kostant, we showed the existence of a 4-form form, in particular, the groups G_2 and $\text{Spin}(7)$.
- The 3-form α was already found* by C. Chevalley in [8].
- One can link this work to [1], [7], and [4].

* This is disputed by R. Bryant (cf. [6]). Indeed, the existence of a 3-form stabilised by G_2 was known for at least sixty years before this paper was written, and Chevalley does not appear to construct such a 3-form in his book.

References

- [1] Andre Lichnerowicz. *Généralisations de la géométrie kählérienne globale*. French. 1951 (cit. on p. 5).
- [2] Marcel Berger. “Sur les groupes d’holonomie homogènes de variétés à connexion affine et des variétés riemanniennes”. In: *Bulletin de la Société Mathématique de France* 83 (1955), pp. 279–330. DOI: 10.24033/bsmf.1464 (cit. on p. 1).
- [3] Bertram Kostant. “On Invariant Skew-Tensors”. In: *Proceedings of the National Academy of Sciences* 42.3 (1956), pp. 148–151. DOI: 10.1073/pnas.42.3.148. JSTOR: 88918 (cit. on p. 5).
- [4] André Weil. “Un théorème fondamental de Chern en géométrie riemannienne”. In: *Séminaire Bourbaki: année 1961/62, exposés 223-240*. Séminaire Bourbaki 7. 1962, pp. 273–284 (cit. on p. 5).
- [7] Shiing-shen Chern. “On a Generalization of Kähler Geometry”. In: *Algebraic Geometry and Topology. A Symposium in Honor of Solomon Lefschetz*. JSTOR: j.ctt183pmq2.10 (cit. on p. 5).
- [8] Claude C. Chevalley. *The Algebraic Theory of Spinors* (cit. on p. 5).